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Density dependence of a Zufiria-type model for Rayleigh–Taylor bubble fronts

Sung-Ik Sohn*

Department of Mathematics, Kangnung National University, Kangnung 210-702, Korea

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We present the analytic model for the evolution of bubbles of arbitrary density ratio in Rayleigh–Taylor and Richtmyer–Meshkov instabilities. The model is the generalization of Zufiria's potential theory, which is based on the velocity potential with a point source and previously applied only for the interface of infinite density ratio. The analytic expressions for asymptotic solutions of bubbles are obtained. The predictions from the Zufiria model agree well with the numerical results not only for the bubble velocity, but also for the bubble curvature. It is found that the asymptotic curvature of a Richtmyer–Meshkov bubble is smaller than that of a Rayleigh–Taylor bubble for all Atwood numbers.

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The phenomenon of unstable interfacial fluid mixing occurs frequently in basic sciences and engineering applications. A gravity-driven interfacial instability is known as the Rayleigh–Taylor (RT) instability [1] and a shock-driven interfacial instability is known as the Richtmyer–Meshkov (RM) instability [2]. Both instabilities play important roles in many fields ranging from astrophysics to inertial confinement fusion, and are subjects of intensive current research. A wide range of literature on these fields is available, and can be found and traced in Refs. [3–15].

Small perturbations at these unstable interfaces grow into nonlinear structures in the form of bubbles and spikes [3]. A bubble (spike) is a portion of the light (heavy) fluid penetrating into the heavy (light) fluid. At later times, a bubble in the RT instability attains a constant velocity, while a RM bubble has a decaying growth rate. Eventually, a turbulent mixing caused by vortex structures around spikes breaks the ordered fluid motion.

The theoretical models for comprehensive descriptions of the motion of bubbles at unstable interfaces are potential flow models proposed by Layzer [4] and Zufiria [5]. Both Layzer and Zufiria models approximate the shape of the interface near the bubble tip as a parabola and give a set of ordinary differential equations to determine the position, velocity, and curvature of the bubble. The main difference between the two models is that the velocity potential in the Layzer model is an analytical function of sinusoidal form, while in the Zufiria model, it has a point source (singularity) and is derived from the complex conformal mapping.

Since Layzer [4] proposed the model for RT instability of infinite density ratio, it has been studied by many people [7–10,13–15] and recently extended to the system of finite density ratios [13–15]. However, the Zufiria-type model has been less developed than the Layzer model, due to the sophisticated form of velocity potential, and so far, is limited to the case of infinite density ratio [12]. In this communication, we generalize the Zufiria model to the unstable interface of arbitrary density ratio and obtain analytic expressions for asymptotic solutions for bubbles in RT and RM instabilities.

We show a surprising result that the predictions from the Zufiria model for asymptotic curvatures of a RT bubble and a RM bubble are different from each other and are quantitatively larger than the results of the Layzer model [13,14]. This raises the issue for the validity of modelings, and therefore direct comparisons of the solutions of the models with the numerical results are presented.

Note that the solution of single-mode bubbles at unstable interfaces not only has its own fundamental importance, but also is a key factor in the dynamics of the bubble merger in the evolution of multimode interfaces [6,8].

We consider an interface in a vertical channel filled with two fluids of different densities in two dimensions. The density of upper and lower fluids is denoted as ρ_1 and ρ_2, respectively. From the assumption of potential flows, there exist complex potentials $W_1(z)=\phi_1+i\psi_1$ for the upper fluid and $W_2(z)=\phi_2+i\psi_2$ for the lower fluid, where ϕ is the velocity potential and ψ the stream function. In the laboratory frame of reference, the location of the bubble tip is $Z(t)=X(t)+iY(t)$ with $Y(t)=L/2$, where L is a channel width. The bubble moves in the x direction with tip velocity U. It is convenient to choose a frame of reference $(\hat{x},\hat{y})$ moving together with the tip of the bubble. In other words, the frame of reference moves with the bubble velocity U. In this moving frame, the location of the bubble tip is $\hat{x}=\hat{y}=0$ and the interface near the bubble tip is approximated as

$$\eta(\hat{x},\hat{y},t) = \hat{y}^2 + 2R(t)\hat{x} = 0, \tag{1}$$

where R is the local radius of curvature.

The evolution of bubble can be determined by the kinematic condition

$$\frac{D\eta(\hat{x},\hat{y},t)}{Dt} = 2\frac{dR}{dt}\hat{x} + 2Ru + 2\hat{y}v = 0, \tag{2}$$

and the Bernoulli equation

*Electronic address: sohnsi@kangnung.ac.kr

$$\rho_1\left[\frac{\partial\phi_1}{\partial t}+\frac{1}{2}(\nabla\phi_1)^2+\left(g+\frac{dU}{dt}\right)\hat{x}\right]=\rho_2\left[\frac{\partial\phi_2}{\partial t}+\frac{1}{2}(\nabla\phi_2)^2+\left(g+\frac{dU}{dt}\right)\hat{x}\right], \quad (3)$$

where u and v are $\hat{x}$ and $\hat{y}$ components of the interface velocity, and g is an external acceleration.

Extending Zufiria's model, we take complex potentials

$$W_1(\hat{z})=Q_1\log[1-e^{-k(\hat{z}+H)}]-U\hat{z}, \quad (4)$$

$$W_2(\hat{z})=Q_2\log[1-e^{-k(\hat{z}-H)}]+(K-U)\hat{z}, \quad (5)$$

where $k=2\pi/L$ is the wave number. Note that $\phi_1\to -U\hat{x}$ as $\hat{x}\to+\infty$, and $\phi_2\to e^{-kH}(K-U)\hat{x}$ as $\hat{x}\to-\infty$.

Expanding Eqs. (4) and (5) in powers of $\hat{z}$, we have

$$W_1=Q_1\sum_{i=0}^{\infty}\frac{c_i}{i!}\hat{z}^i-U\hat{z}, \quad (6)$$

$$W_2=Q_2\sum_{i=0}^{\infty}\frac{\tilde{c}_i}{i!}\hat{z}^i+(K-U)\hat{z}. \quad (7)$$

The expressions for $c_i(H)$ are given in Ref. [12] and $\tilde{c}_i(H)=c_i(-H)$. The relation $dW_i/d\hat{z}=u-iv$, $i=1$, 2, gives the expressions for the interface velocity taken from the upper and lower fluids. Substituting these expressions into Eq. (2) and satisfying up to first order in $\hat{x}$, it gives

$$U=c_1Q_1=\tilde{c}_1Q_2+K, \quad (8)$$

$$\frac{dR}{dt}=-Q_1(3c_2+c_3R)R=-Q_2(3\tilde{c}_2+\tilde{c}_3R)R. \quad (9)$$

Using Eqs. (4) and (5), the first- and second-order equations in $\hat{x}$ of Eq. (3) are

$$(c_1+c_2R)\frac{dQ_1}{dt}+Q_1(c_2+c_3R)\frac{dH}{dt}-Q_1^2c_2^2R+g=\frac{1-A}{1+A}\left[(\tilde{c}_1+\tilde{c}_2R)\frac{dQ_2}{dt}+\frac{dK}{dt}+Q_2(\tilde{c}_2+\tilde{c}_3R)\frac{dH}{dt}-Q_2^2\tilde{c}_2^2R+g\right], \quad (10)$$

$$\left(\frac{c_2}{2}+c_3R+c_4\frac{R^2}{6}\right)\frac{dQ_1}{dt}+Q_1\left(\frac{c_3}{2}+c_4R+c_5\frac{R^2}{6}\right)\frac{dH}{dt}+\frac{Q_1^2}{2}F_1=\frac{1-A}{1+A}\left[\left(\frac{\tilde{c}_2}{2}+\tilde{c}_3R+\tilde{c}_4\frac{R^2}{6}\right)\frac{dQ_2}{dt}+Q_2\left(\frac{\tilde{c}_3}{2}+\tilde{c}_4R+\tilde{c}_5\frac{R^2}{6}\right)\frac{dH}{dt}+\frac{Q_2^2}{2}F_2\right], \quad (11)$$

where

$$F_1=c_2^2-2c_2c_3R+(3c_3^2-4c_2c_4)\frac{R^2}{3},$$

$$F_2=\tilde{c}_2^2-2\tilde{c}_2\tilde{c}_3R+(3\tilde{c}_3^2-4\tilde{c}_2\tilde{c}_4)\frac{R^2}{3},$$

and $A=(\rho_1-\rho_2)/(\rho_1+\rho_2)$ represents the Atwood number. Equations (8)–(11) determine the dynamics of the bubbles of finite density contrast.

Sohn and Zhang [12] showed that, for $A=1$, the linear theory for Zufiria's model, in small amplitudes or early times, agrees with the result of the linearized Euler equations for both RT and RM instabilities. The linear theory for Zufiria's model for $A=1$ can be directly extended to the general case of $A<1$, using Eqs. (8)–(11). The details of its derivation will be given elsewhere.

We now find the asymptotic solutions for bubbles. For a bubble in RT instability, all time derivatives of variables in Eqs. (9)–(11) converge to zero at a later time. Then, from Eq. (9), we have

$$3c_2+c_3R\to 0 \quad \text{and} \quad Q_2\to 0, \quad (12)$$

and Eqs. (10) and (11) reduce to

$$(Q_1c_2)^2R\to\frac{2A}{1+A}g \quad \text{and} \quad F_1\to 0. \quad (13)$$

Solving these equations with Eq. (8), the asymptotic solution for a bubble of RT instability is

$$R\to\frac{\sqrt{3}}{k},\ H\to\frac{1}{k}\ln(2+\sqrt{3}),\ Q_1\to\frac{2}{3^{1/4}}\sqrt{\frac{2Ag}{(1+A)k^3}},$$

$$U\to\frac{\sqrt{6+4\sqrt{3}}}{2+\sqrt{3}}\sqrt{\frac{2Ag}{3(1+A)k}},\ Q_2\to 0,\ K\to U. \quad (14)$$

The functional form of the asymptotic bubble velocity in Eq. (14) is similar to the solution of the Layzer model, obtained by Goncharov [13], except the factor $\sqrt{6+4\sqrt{3}}/(2+\sqrt{3})=0.963$. On the other hand, the asymptotic solutions for the bubble curvature of two models have a large quantitative difference. Denoting the bubble curvature as $\xi=1/R$, the solution of the Layzer model is $\xi_{\mathrm{Layzer}}\to k/3$, while in the Zufiria model, $\xi_{\mathrm{Zufiria}}\to k/\sqrt{3}$ from Eq. (14).

The analytic solutions of the models are validated by comparing with numerical results. In fact, the numerical results for the bubble curvature of finite density ratio are very rare. The author [16] recently performed the numerical simulations for RT-type instability by the vortex method and reported the results, including the bubble curvature, for several Atwood numbers. Table I is the comparison of the numerical results for the asymptotic velocity of a RT bubble in Ref.

TABLE I. Asymptotic bubble velocities of RT instability.

A	U_{num}	U_{Zufiria}	U_{Layzer}
0.05	0.169	0.172	0.178
0.3	0.370	0.378	0.392
0.7	0.500	0.505	0.524
1.0	0.540	0.556	0.577

TABLE II. Asymptotic bubble curvatures of RT instability.

A	$\xi_{\rm num}$	$\xi_{\rm Zufiria}$	$\xi_{\rm Layzer}$
0.05	0.506 ± 0.002	0.577	0.333
0.3	0.536 ± 0.004	0.577	0.333
0.7	0.545 ± 0.006	0.577	0.333
1.0	0.509 ± 0.006	0.577	0.333

[16] with the theoretical predictions from Zufiria's model and Layzer's model for the Atwood numbers, $A=0$, 0.3, 0.7, and 1. In Table I, the velocity is scaled by $\sqrt{g/k}$ and has a dimensionless unit. We see that the predictions from the Zufiria model are in excellent agreement with the numerical results for all cases. In Table II, we compare the theoretical predictions for the asymptotic curvature of a RT bubble from two models with the numerical results in Ref. [16]. The curvature in Table II is dimensionless, scaled by the wave number k. The numerical solutions for the bubble curvature converge to limits between 0.5 and 0.55. Table II shows that the Zufiria model provides a good prediction for the bubble curvature, while the prediction from the Layzer model is too small for the numerical results.

For a bubble in RM instability, g is set to 0. At a later time, the bubble velocity can be expressed as $U\sim\epsilon/t^{\alpha}$. From Eq. (8), we have $Q_1\sim\delta/t^{\alpha}$ with $c_1\delta=\epsilon$. From Eqs. (9)–(11), one concludes that α is equal to 1, and $Q_2\sim\tau/t^{\beta}$, $\beta>1$, and $K\sim\epsilon/t$, using the fact that dR/dt and dH/dt have to go to zero faster than $1/t$. Then, at late time, Eqs. (9)–(11) become

$$3c_2+c_3R\rightarrow 0, \tag{15}$$

$$(c_1+c_2R)+\delta c_2^2R-\frac{1-A}{1+A}c_1\rightarrow 0, \tag{16}$$

$$c_2+2c_3R+\frac{1}{3}c_4R^2-\delta F_1\rightarrow 0. \tag{17}$$

Finding the expression for δ from Eq. (16) with Eq. (15) and substituting it to Eq. (17), we obtain a cubic polynomial

$$(3-A)\lambda^3-(21+9A)\lambda^2+(3+15A)\lambda-4A=0, \tag{18}$$

where $\lambda=e^{kH(t\rightarrow\infty)}$. The polynomial (18) has one real root for $A>0.0376$. For $0\leqslant A\leqslant 0.0376$, it has three real roots, but only one of them is larger than 1, which is taken for our solution. The analytic expression of the solution for Eq. (18) is lengthy and is not given here. The solution for Eq. (18) is an increasing function with respect to A, having the values, 6.85 at $A=0$ and 14.38 at $A=1$. Then, the asymptotic solution for a bubble in RM instability is

$$R\rightarrow\frac{3(\lambda-1)}{k(\lambda+1)},\quad H\rightarrow\frac{1}{k}\ln\lambda,\quad Q_1\sim\frac{(\lambda-1)^2((A+3)\lambda-2A)}{3(1+A)\lambda^2k^2t},$$

$$U\sim\left(\frac{A+3}{3(1+A)}-\frac{1}{\lambda}+\frac{2A}{3(1+A)\lambda^2}\right)\frac{1}{kt},\quad K\rightarrow U. \tag{19}$$

The solution for the asymptotic bubble velocity in Eq. (19) has the correction terms, $-1/\lambda+2A/(3(1+A)\lambda^2)$, to the solution of the Layzer model [13]. The coefficient of $1/kt$ in the expression of the asymptotic velocity in Eq. (19) decreases from 0.854 at $A=0$ to 0.599 at $A=1$. Thus, the prediction for the asymptotic growth rate from the Zufiria model is about 15% smaller than that from the Layzer model for $A=0$ and 10% smaller for $A=1$.

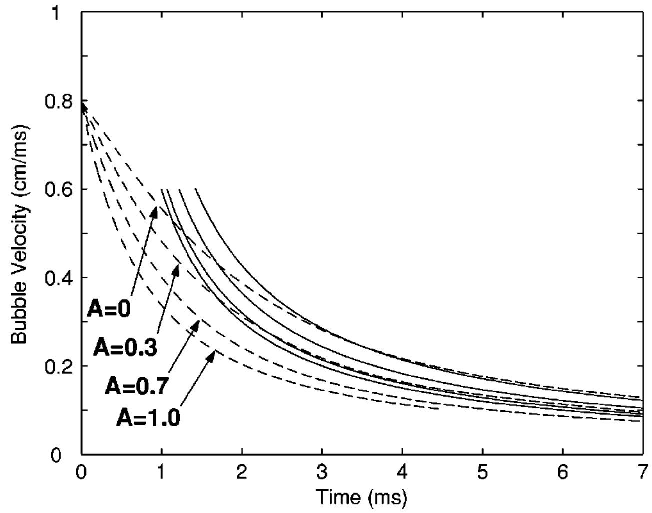

FIG. 1. Bubble velocities of RM instability. The solid curves are the theoretical predictions from Zufiria's model for the asymptotic growth rate of bubble for $A=0$, 0.3, 0.7, and 1 from above to below. The dashed curves are the numerical results in Ref. [16].

In the Zufiria model, the asymptotic bubble curvature of RM instability is different from that of RT instability, while the Layzer model gives the same solution $\xi_{\rm Layzer}\rightarrow k/3$ for both instabilities. Moreover, the asymptotic curvature of a RM bubble in the Zufiria model depends on the Atwood number and is smaller than that of a RT bubble for all A. Note that the bubble curvature of RM instability, $\xi_{\rm Zufiria}\rightarrow k(\lambda+1)/(3(\lambda-1))$, is a decreasing function with respect to A, since $\lambda(A)$ is an increasing one.

In Fig. 1, we compare the asymptotic solution for the bubble velocity of RM instability from the Zufiria model with the numerical results [16] for the Atwood numbers, $A=0$, 0.3, 0.7, and 1. The solid curves in Fig. 1 come from the asymptotic solution (19), not from the numerical solution of Eqs. (8)–(11). Figure 1 shows that the growth rates of the RM bubble decay to zero for all cases and the decaying rate is faster for a larger Atwood number. The predictions from the Zufiria model fit well with the numerical results at late time. Although it is not plotted in Fig. 1, the predictions from the Layzer model for the asymptotic growth rate are slightly larger than the solutions of the Zufiria model, as mentioned above. Table III is the comparison of the numerical results for the asymptotic curvature of a RM bubble with the theoretical predictions from two models. The curvature in Table III has a dimensionless unit, similarly as in Table II. The predictions from the Zufiria model are in relatively good agreement with the numerical results, which converge to limits between 0.45 and 0.5. From Table III, we observe that, in the Zufiria model, the asymptotic bubble curvatures of RM instability are smaller than the RT cases for all Atwood num-

TABLE III. Asymptotic bubble curvatures of RM instability.

A	ξ_{num}	ξ_{Zufiria}	ξ_{Layzer}
0.0	0.451 ± 0.002	0.447	0.333
0.3	0.484 ± 0.005	0.423	0.333
0.7	0.491 ± 0.009	0.398	0.333
1.0	0.464 ± 0.016	0.383	0.333

bers and this behavior is in accordance with the numerical results.

In summary, the Zufiria-type model has been extended to the unstable interfaces of finite density ratio and the asymptotic solutions of bubbles are obtained for RT and RM instabilities. The predictions for asymptotic bubble curvatures, as well as asymptotic bubble velocities, from Zufiria's model agree better with the numerical results than Layzer's model. The quantitative differences between the two models for the predictions of solutions come from the choice of potentials, and the Zufiria-type potentials are more appropriate for the description of unstable interfaces than the Layzer-type potentials. The Zufiria model also theoretically validates the recent numerical result that the asymptotic curvature for a RM bubble is smaller than that for a RT bubble for all Atwood numbers. Therefore, we conclude that, at late time, the bubble front of RM instability is larger than that of RT instability.

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